

On the asymptotically simplicity of periodic eigenvalues and Titchmarsh's formula

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Abstract

We consider Sturm-Liouville equation $y'' + (\lambda - q)y = 0$ where $q \in L^1[0, a]$. We obtain various conditions on the Fourier coefficients of q such that the periodic eigenvalues having the form given by Titchmarsh are asymptotically simple. Under these conditions, we give some asymptotic estimates for the spectral gaps.

Key words: Periodic eigenvalues; Simple eigenvalues; Spectral gaps

1 Introduction

In this study we consider Hill's equation

$$y'' + (\lambda - q(x))y = 0, \quad (1.1)$$

where $q(x)$ is a real-valued summable function on $[0, a]$ for $a > 0$ extended to the real line by periodicity. Let the points $\lambda_{2m+1}, \lambda_{2m+2}$ denote the periodic eigenvalues of (1.1) on the interval $[0, a]$ with the periodic boundary conditions

$$y(0) = y(a), \quad y'(0) = y'(a). \quad (1.2)$$

The number of periodic eigenvalues is countably infinite and the eigenvalues form a monotone increasing sequence with a single accumulation point at ∞ . See basics and further references in [1,2,3,4].

In the classical investigations, the order of the asymptotic estimates for the two eigenvalues $\lambda_{2m+1}, \lambda_{2m+2}$ is closely related to the order of smoothness

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of the potential q . We mention in particular [1,3,5,6] and the latest results in [7]. In [1], Theorem 4.2.4, if, for $r \geq 1$, q has an absolutely continuous $(r-1)$ st derivative on $(-\infty, \infty)$, the asymptotic estimates for the $\lambda_{2m+1}, \lambda_{2m+2}$ as $m \rightarrow \infty$ have an error term of order $o(m^{-r})$.

In the case when $r = 0$, q is assumed to be piecewise continuous with period a . The presented proof of Theorem 4.2.4 in [1] is due to Hochstadt [5] by using the Prüfer transformation. Then Titchmarsh [3] (see Section 21.5) refined the o -term in [1], which leads to the following formulas for the $\lambda_{2m+1}, \lambda_{2m+2}$, depending on the Floquet theory (see [1], Chapter 2),

$$\frac{\lambda_{2m+2}}{\lambda_{2m+1}} = \frac{(2m+2)^2 \pi^2}{a^2} \pm |c_{2m+2}| + O(m^{-1/2}) \quad (1.3)$$

as $m \rightarrow \infty$, where

$$c_{2m+2} =: (q, e^{i2(2m+2)\pi x/a}) =: \frac{1}{a} \int_0^a q(x) e^{-i2(2m+2)\pi x/a} dx$$

is the Fourier coefficient of q and without loss of generality we always suppose that $c_0 = 0$. Note that the O -term in (1.3) was essentially given as $O(m^{-1})$, but Eastham [1] explained, Theorem 4.4.1, that Titchmarsh's proof produces only the O -term as in (1.3). For the first time, by using the Prüfer transformation, Brown and Eastham [7] proved, Theorem 2.2, that if q is locally integrable on $(-\infty, \infty)$, then the Titchmarsh's formula (1.3) can indeed be improved to the O -term $O(m^{-1})$. The present work was stimulated by the papers [7,8,9].

By using a perturbation method developed in [8,10,11] we obtain that, under the hypotheses of Theorem 2 (see also (1.6) and (1.7)), the O -term in (1.3) can indeed be improved to the O -term $O(\rho(m) m^{-1})$ which also implies $o(m^{-1})$ and all the periodic eigenvalues are asymptotically simple. In Section 4, using these estimates, the widths of instability intervals are given with the isolated Fourier coefficients of q .

Now, let us formulate the subsequent form of the Riemann-Lebesgue lemma. Since the proof of lemma repeats the arguments of the Lemma 6 in [9], we omit its proof.

Lemma 1 *If $q \in L^1[0, a]$ then $\int_0^x q(t) e^{i2m\pi t/a} dt \rightarrow 0$ as $|m| \rightarrow \infty$ uniformly in x .*

Using this, we define

$$\rho(m) =: \sup_{0 \leq x \leq a} \left| \int_0^x q(t) e^{\mp i2(2m+2)\pi t/a} dt \right| \quad (1.4)$$

and then we get $\rho(m) \rightarrow 0$ as $m \rightarrow \infty$.

In this paper, we prove the following main result:

Theorem 2 *Let q be locally integrable on $(-\infty, \infty)$ and assume that the condition*

$$\lim_{m \rightarrow \infty} \frac{\rho(m)}{m c_{2m+2}} = 0 \quad (1.5)$$

holds. Then the large periodic eigenvalues are simple and (1.3) holds with the improved O -term $O(\rho(m) m^{-1})$, where $\rho(m)$, defined in (1.4), is an order of the Fourier coefficient of q .

Clearly, in Theorem 2, if, instead of (1.5), we assume that either the condition

$$c_{2m+2} \sim \rho(m), \quad (1.6)$$

where the notation $a_m \sim b_m$ means that there exist constants c_1, c_2 such that $0 < c_1 < c_2$ and $c_1 < |a_m/b_m| < c_2$ for all sufficiently large m , or the condition

$$|c_{2m+2}| > \varepsilon m^{-1} \quad \text{with some } \varepsilon > 0 \quad (1.7)$$

holds, then the assertion of Theorem 2 remains valid.

It easily follows from [1], Theorem 4.2.3, that the periodic eigenvalues $\lambda_{2m+1}, \lambda_{2m+2}$ are asymptotically located in pairs, satisfying the following asymptotic estimate

$$\lambda_{2m+1} = \lambda_{2m+2} + o(1) = (2m+2)^2 \pi^2 / a^2 + o(1), \quad (1.8)$$

for sufficiently large m . This estimate implies that the pair of the eigenvalues $\{\lambda_{2m+1}, \lambda_{2m+2}\}$ is close to the number $(2m+2)^2 \pi^2 / a^2$ and isolated from the remaining spectrum of the problem by a distance of size m . In particular, by using (1.8), the following inequality holds

$$|\lambda_{m,j} - \frac{(2(m-k)+2)^2 \pi^2}{a^2}| > a^{-2} |k| |(2m+2) - k| > C m, \quad (1.9)$$

for all $k \neq 0, (2m+2), j = 1, 2$ and $k \in \mathbb{Z}$, where, here and in subsequent relations, we denote λ_{2m+1} and λ_{2m+2} by $\lambda_{m,1}$ and $\lambda_{m,2}$ respectively for sufficiently large m and C is positive constant whose exact value is not essential. For $q = 0$, $\{e^{-i(2m+2)\pi x/a}, e^{i(2m+2)\pi x/a}\}$ is a basis of the eigenspace corresponding to the eigenvalue $(2m+2)^2 \pi^2 / a^2 \neq 0$ of the problem (1.1)-(1.2) on $[0, a]$.

2 Preliminaries

Let us consider the following relation, for sufficiently large m , in order to obtain the values of periodic eigenvalues $\lambda_{2m+1}, \lambda_{2m+2}$ corresponding to the

normalized eigenfunctions $\Psi_{m,1}(x), \Psi_{m,2}(x)$:

$$\Lambda_{m,j,m-k}(\Psi_{m,j}, e^{i(2(m-k)+2)\pi x/a}) = (q \Psi_{m,j}, e^{i(2(m-k)+2)\pi x/a}), \quad (2.1)$$

where $\Lambda_{m,j,m-k} = (\lambda_{m,j} - (2(m-k) + 2)^2 \pi^2 / a^2)$, $j = 1, 2$. This relation can be obtained from the equation (1.1), considering $\Psi_{m,j}(x)$ instead of y and multiplying both sides by $e^{i(2(m-k)+2)\pi x/a}$. Moreover, to iterate (2.1) we use the following relations (e.g., see Lemma 1 in [12]),

$$(q \Psi_{m,j}, e^{i(2m+2)\pi x/a}) = \sum_{m_1=-\infty}^{\infty} c_{m_1}(\Psi_{m,j}, e^{i(2(m-m_1)+2)\pi x/a}), \quad (2.2)$$

$$|(q \Psi_{m,j}, e^{i(2(m-m_1)+2)\pi x/a})| < 3M \quad (2.3)$$

for all sufficiently large m , where $m_1 \in \mathbb{Z}$, $j = 1, 2$ and $M = \sup_{m \in \mathbb{Z}} |c_m|$.

Now, using (2.2) in (2.1) for $k = 0$ and then isolating the terms with indices $m_1 = 0, (2m + 2)$, we get

$$\begin{aligned} \Lambda_{m,j,m}(\Psi_{m,j}, e^{i(2m+2)\pi x/a}) &= c_{2m+2}(\Psi_{m,j}, e^{-i(2m+2)\pi x/a}) + \\ &\quad \sum_{m_1 \neq 0, (2m+2)} c_{m_1}(\Psi_{m,j}, e^{i(2(m-m_1)+2)\pi x/a}) \end{aligned} \quad (2.4)$$

by the assumption $c_0 = 0$. After using (2.1) for $k = m_1$ in (2.4), again using (2.2) with a suitable indices we get the following relation

$$[\Lambda_{m,j,m} - a(\lambda_{m,j})]u_{m,j} = [c_{2m+2} + b(\lambda_{m,j})]v_{m,j} + R(m), \quad (2.5)$$

where $j = 1, 2$,

$$u_{m,j} = (\Psi_{m,j}, e^{i(2m+2)\pi x/a}), \quad v_{m,j} = (\Psi_{m,j}, e^{-i(2m+2)\pi x/a}), \quad (2.6)$$

$$\begin{aligned} a(\lambda_{m,j}) &= \sum_{m_1} \frac{c_{m_1} c_{-m_1}}{\Lambda_{m,j,m-m_1}}, \quad b(\lambda_{m,j}) = \sum_{m_1} \frac{c_{m_1} c_{2m+2-m_1}}{\Lambda_{m,j,m-m_1}}, \\ R(m) &= \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2} (q \Psi_{m,j}, e^{i(2(m-m_1-m_2)+2)\pi x/a})}{\Lambda_{m,j,m-m_1} \Lambda_{m,j,m-m_1-m_2}}. \end{aligned} \quad (2.7)$$

The sums in these formulas are taken over all integers m_1, m_2 such that $m_1, m_2, m_1 + m_2 \neq 0$ and $m_1, m_1 + m_2 \neq 2m + 2$.

Similarly, by considering the other eigenfunction $e^{-i(2m+2)\pi x/a}$ corresponding to the eigenvalue $(2m+2)^2 \pi^2 / a^2$ of the problem (1.1)-(1.2) for $q = 0$, one can easily obtain the following relation

$$[\Lambda_{m,j,m} - a'(\lambda_{m,j})]v_{m,j} = [c_{-2m-2} + b'(\lambda_{m,j})]u_{m,j} + R'(m). \quad (2.8)$$

where

$$a'(\lambda_{m,j}) = \sum_{m_1} \frac{c_{m_1} c_{-m_1}}{\Lambda_{m,j,m+m_1}}, \quad b'(\lambda_{m,j}) = \sum_{m_1} \frac{c_{m_1} c_{-2m-2-m_1}}{\Lambda_{m,j,m+m_1}},$$

$$R'(m) = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2} (q \Psi_{m,j}, e^{i(2(m+m_1+m_2)+2)\pi x/a})}{\Lambda_{m,j,m+m_1} \Lambda_{m,j,m+m_1+m_2}}, \quad (2.9)$$

and the sums in these formulas are taken over all integers m_1, m_2 such that $m_1, m_2, m_1 + m_2 \neq 0$ and $m_1, m_1 + m_2 \neq -2m - 2$.

By using (1.9), (2.1) and (2.3), we get (see [12, Theorem 2])

$$\sum_{k \in \mathbb{Z}; k \neq \pm(m+1)} \left| (\Psi_{m,j}, e^{i2k\pi x/a}) \right|^2 = O\left(\frac{1}{m^2}\right)$$

and by using the equality

$$\frac{1}{m_1(2m+2-m_1)} = \frac{1}{2m+2} \left(\frac{1}{m_1} + \frac{1}{2m+2-m_1} \right),$$

we can easily be shown the following relation

$$\sum_{m_1 \neq 0, (2m+2)} \frac{1}{|m_1| |2m+2-m_1|} = O\left(\frac{\ln|m|}{m}\right). \quad (2.10)$$

Therefore, we obtain that the normalized eigenfunctions $\Psi_{m,j}(x)$ have an expansion in terms of the orthonormal basis $\{e^{i2k\pi x/a} : k \in \mathbb{Z}\}$ on $[0, a]$ of the form (see also related result in (78) on p. 77 of [6])

$$\Psi_{m,j}(x) = u_{m,j} e^{i(2m+2)\pi x/a} + v_{m,j} e^{-i(2m+2)\pi x/a} + h_m(x), \quad (2.11)$$

where

$$(h_m, e^{\mp i(2m+2)\pi x/a}) = 0, \quad \|h_m\| = O(m^{-1}), \quad \sup_{x \in [0, a]} |h_m(x)| = O\left(\frac{\ln|m|}{m}\right) \quad (2.12)$$

$$|u_{m,j}|^2 + |v_{m,j}|^2 = 1 + O(m^{-2}). \quad (2.13)$$

3 Estimates for the eigenvalues

The following estimates play an essential role in the proof of main result of the paper.

Lemma 3 *The eigenvalues $\lambda_{2m+1}, \lambda_{2m+2}$ of the problem (1.1)-(1.2) satisfy, for $m \geq N$,*

$$\lambda_{2m+1}, \lambda_{2m+2} = (2m+2)^2 \pi^2 / a^2 + O(\rho(m)), \quad (3.1)$$

where $\rho(m)$ is defined in (1.4).

PROOF. From the relations (1.9) and (2.10), one can easily see that

$$\begin{aligned} & \sum_{m_1 \neq 0, \pm(2m+2)} \left| \frac{1}{\Lambda_{m,j,m \mp m_1}} - \frac{1}{\Lambda_{m,m \mp m_1}^0} \right| \\ & \leq C \frac{|\Lambda_{m,j,m}|}{m} \sum_{m_1 \neq 0, \pm(2m+2)} \frac{1}{|m_1||2m+2 \mp m_1|} = O\left(\frac{\Lambda_{m,j,m}}{m}\right), \end{aligned} \quad (3.2)$$

where $\Lambda_{m,m \mp m_1}^0 = ((2m+2)^2\pi^2/a^2 - (2(m \mp m_1) + 2)^2\pi^2/a^2)$. Thus, we get

$$a(\lambda_{m,j}) = \frac{a^2}{4\pi^2} \sum_{m_1 \neq 0, (2m+2)} \frac{c_{m_1} c_{-m_1}}{m_1(2m+2-m_1)} + O\left(\frac{\Lambda_{m,j,m}}{m}\right). \quad (3.3)$$

It also follows from [13] (see Lemma 2) that, in our notations,

$$\begin{aligned} a(\lambda_{m,j}) &= \frac{a^2}{2\pi^2} \sum_{m_1 > 0, m_1 \neq (2m+2)} \frac{c_{m_1} c_{-m_1}}{(2m+2+m_1)(2m+2-m_1)} + O\left(\frac{\Lambda_{m,j,m}}{m}\right) \\ &= a^2 \int_0^a (G^+(x, m) - G_0^+(m))^2 e^{i2(4m+4)\pi x/a} dx + O\left(\frac{\Lambda_{m,j,m}}{m}\right), \end{aligned} \quad (3.4)$$

where

$$G_{m_1}^\pm(m) =: (G^\pm(x, m), e^{i2m_1\pi x/a}) = \frac{c_{m_1 \pm (2m+2)}}{i2\pi m_1} \quad (3.5)$$

for $m_1 \neq 0$ and $G_0^\pm(m) =: (G^\pm(x, m), 1)$ for $m_1 = 0$ are the Fourier coefficients of the functions

$$G^\pm(x, m) = \frac{1}{a} \int_0^x q(t) e^{\mp i2(2m+2)\pi t/a} dt - \frac{1}{a} c_{\pm(2m+2)} x \quad (3.6)$$

with respect to the trigonometric system $\{e^{i2m_1\pi x/a} : m_1 \in \mathbb{Z}\}$ and

$$G^\pm(x, m) - G_0^\pm(m) = \sum_{m_1 \neq (2m+2)} \frac{c_{m_1}}{i2\pi(m_1 \mp (2m+2))} e^{i2(m_1 \mp (2m+2))\pi x/a}. \quad (3.7)$$

Now, taking into account the equalities (1.4) and (3.6), we obtain the estimates

$$G^\pm(x, m) - G_0^\pm(m) = G^\pm(x, m) - \frac{1}{a} \int_0^a G^\pm(x, m) dx = O(\rho(m)), \quad (3.8)$$

$$G^\pm(a, m) = G^\pm(0, m) = 0.$$

Then, these with integration by parts and $q \in L^1[0, a]$ imply that (see (3.4))

$$a(\lambda_{m,j}) = \frac{-a^2}{i2\pi(4m+4)} \times$$

$$\int_0^a 2(G^+(x, m) - G_0^+(m))(q(x)e^{-i2(2m+2)\pi x/a} - c_{2m+2})e^{i2(4m+4)\pi x/a} dx + O\left(\frac{\Lambda_{m,j,m}}{m}\right)$$

$$= O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{\Lambda_{m,j,m}}{m}\right) \quad (3.9)$$

for sufficiently large m .

Similarly, by virtue of (3.2) we get

$$\begin{aligned} b(\lambda_{m,j}) &= \frac{a^2}{4\pi^2} \sum_{m_1 \neq 0, (2m+2)} \frac{c_{m_1} c_{2m+2-m_1}}{m_1(2m+2-m_1)} + O\left(\frac{\Lambda_{m,j,m}}{m}\right) \\ &= -a^2 \int_0^a (Q(x) - Q_0)^2 e^{-i2(2m+2)\pi x/a} dx + O\left(\frac{\Lambda_{m,j,m}}{m}\right) \\ &= \frac{-a^2}{i2\pi(2m+2)} \int_0^a 2(Q(x) - Q_0) q(x) e^{-i2(2m+2)\pi x/a} dx + O\left(\frac{\Lambda_{m,j,m}}{m}\right), \end{aligned} \quad (3.10)$$

where $Q(x) = a^{-1} \int_0^x q(t) dt$, $Q_{m_1} =: (Q(x), e^{i2m_1\pi x/a}) = \frac{c_{m_1}}{i2\pi m_1}$ if $m_1 \neq 0$,

$$Q(x) - Q_0 = \sum_{m_1 \neq 0} Q_{m_1} e^{i2m_1\pi x/a}. \quad (3.11)$$

Therefore, again using integration by parts the integral in (3.10), $Q(a) = c_0 = 0$ and (1.4), we obtain

$$b(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{\Lambda_{m,j,m}}{m}\right). \quad (3.12)$$

Also

$$b'(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{\Lambda_{m,j,m}}{m}\right). \quad (3.13)$$

Let us prove that $R(m) = O(\rho(m))$ (see (2.7)). First, since $q \in L^1[0, a]$ and is different from 0, there exists $x \in [0, a]$ such that

$$\int_0^x q(t) dt \neq 0 \quad (3.14)$$

and the integral (3.14) is finite for all $x \in [0, a]$. Therefore, multiplying the integrand of (3.14) by $e^{-i2(2m+2)\pi x/a} e^{i2(2m+2)\pi x/a}$ and using integration by parts, we have

$$\sup_{0 \leq x \leq a} \left| \int_0^x q(t) dt \right| \leq C(\rho(m) + m\rho(m)) \leq Cm\rho(m)$$

which implies that, for sufficiently large m ,

$$\rho(m) > C m^{-1} \quad (3.15)$$

with some $C > 0$. Then using (1.9), (2.3) and the relation (2.10), we obtain

$$|R(m)| \leq C \frac{(\ln|m|)^2}{m^2} = O(\rho(m)).$$

Similarly, $R'(m) = O(\rho(m))$.

It may readily be seen by a change the sign of summation indices in the relation for $a'(\lambda_{m,j})$ (see (2.5), (2.8)) that $a(\lambda_{m,j}) = a'(\lambda_{m,j})$. On the other hand, the formula (2.13) gives that either $|u_{m,j}| > 1/2$ or $|v_{m,j}| > 1/2$ for large m . Thus If $|u_{m,j}| > 1/2$, then using (2.5), (3.9) and (3.12) with $R(m) = O(\rho(m))$ we get

$$\Lambda_{m,j,m} \left(1 + O\left(\frac{1}{m}\right)\right) = c_{2m+2} \frac{v_{m,j}}{u_{m,j}} + O(\rho(m)).$$

This with (1.4) implies $\Lambda_{m,j,m} = O(\rho(m))$. If $|v_{m,j}| > 1/2$ then, again by (2.8), (3.9), (3.13) and $R'(m) = O(\rho(m))$, we have (3.1). The lemma is proved. ■

Lemma 4 *For all sufficiently large m , the series in (2.7) and (2.9) have the following estimates*

$$R(m) = O(\rho(m)m^{-1}), \quad R'(m) = O(\rho(m)m^{-1}). \quad (3.16)$$

PROOF. First, we prove that $R(m) = O(\rho(m)m^{-1})$. Arguing as in the proof of (3.3) and using Lemma 3 we get

$$R(m) = \sum_{m_1, m_2} \frac{a^4 2^{-4} \pi^{-4} c_{m_1} c_{m_2} (q \Psi_{m,j}, e^{i(2(m-m_1-m_2)+2)\pi x/a})}{m_1(2m+2-m_1)(m_1+m_2)(2m+2-(m_1+m_2))} + O\left(\frac{\rho(m)}{m^2}\right). \quad (3.17)$$

Then, since the eigenfunction $\Psi_{m,j}(x)$ has expansion (2.11), we have

$$R(m) = \frac{a^4}{(2\pi)^4} (S(m)u_{m,j} + I(m)v_{m,j} + r(m)) + O\left(\frac{\rho(m)}{m^2}\right), \quad (3.18)$$

where $S(m)$, $I(m)$ and $r(m)$ are obtained from the series in (3.17) by replacing the numerator of the fraction in the series with $c_{m_1} c_{m_2} C_{-m_1-m_2}$, $c_{m_1} c_{m_2} C_{2m+2-m_1-m_2}$ and $c_{m_1} c_{m_2} (q h_m, e^{i(2(m-m_1-m_2)+2)\pi x/a})$ respectively.

Now to estimate $R(m)$ it is enough to show that

$$|S(m)| + |I(m)| + |r(m)| = O(\rho(m)m^{-1}), \quad (3.19)$$

since $|u_{m,j}| \leq 1$ and $|v_{m,j}| \leq 1$ (see (2.6)) for the normalized eigenfunctions $\Psi_{m,j}(x)$. By using the summation variable m_2 to represent the previous $m_1 + m_2$ in the formula $S(m)$ which is obtained from the series in (3.17), we get

$$S(m) = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2-m_1} C_{-m_2}}{m_1(2m+2-m_1)m_2(2m+2-m_2)}. \quad (3.20)$$

By using the equality

$$\frac{1}{k(2m+2-k)} = \frac{1}{2m+2} \left(\frac{1}{k} + \frac{1}{2m+2-k} \right),$$

we have

$$S(m) = \frac{1}{(2m+2)^2} \sum_{j=1}^4 S_j, \quad (3.21)$$

where

$$S_1 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_1 m_2}, \quad S_2 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_2 (2m+2-m_1)},$$

$$S_3 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_1 (2m+2-m_2)}, \quad S_4 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{(2m+2-m_1)(2m+2-m_2)}.$$

Now arguing as in the proof of (3.12) (see definition of (3.10)), we get

$$S_1 = -4\pi^2 \int_0^a (Q(x) - Q_0)^2 q(x) dx = O(m\rho(m)). \quad (3.22)$$

Here, the O -term in (3.22) is obtained from (3.15), since the integral in (3.22) is $O(1)$. Moreover, by (3.5)-(3.8) and (3.11), we have the following relations

$$\left. \begin{aligned} S_2 &= -4\pi^2 \int_0^a (Q(x) - Q_0)(G^+(x, m) - G_0^+(m)) q(x) e^{i2(2m+2)\pi x/a} dx = O(\rho(m)), \\ S_3 &= -4\pi^2 \int_0^a (Q(x) - Q_0)(G^-(x, m) - G_0^-(m)) q(x) e^{-i2(2m+2)\pi x/a} dx = O(\rho(m)), \\ S_4 &= 4\pi^2 \int_0^a (G^+(x, m) - G_0^+(m))(G^-(x, m) - G_0^-(m)) q(x) dx = O(\rho(m)). \end{aligned} \right\} \quad (3.23)$$

Thus, in view of (3.21)-(3.23), we obtain $S(m) = O(\rho(m)m^{-1})$. From [8], to estimate $I(m)$, using substitutions $k_1 = m_1$, $k_2 = 2m+2-m_1-m_2$ in the formula $I(m)$ and arguing as in (3.20)-(3.21), one can easily obtain that

$$I(m) = \frac{1}{(2m+2)^2} (I_1 + 2I_2 + I_3), \quad (3.24)$$

where

$$I_1 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2} c_{2m+2-m_1-m_2}}{m_1 m_2}, \quad I_2 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2} c_{2m+2-m_1-m_2}}{m_2 (2m+2-m_1)},$$

$$I_3 = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2} c_{2m+2-m_1-m_2}}{(2m+2-m_1)(2m+2-m_2)}.$$

Again by (3.5)-(3.8), (3.11) and the integration by parts only in I_1 , we get the following estimates

$$\left. \begin{aligned} I_1 &= -4\pi^2 \int_0^a (Q(x) - Q_0)^2 q(x) e^{-i2(2m+2)\pi x/a} dx = O(\rho(m)), \\ I_2 &= 4\pi^2 \int_0^a (Q(x) - Q_0)(G^+(x, m) - G_0^+(m)) q(x) dx = O(\rho(m)), \\ I_3 &= -4\pi^2 \int_0^a (G^+(x, m) - G_0^+(m))^2 q(x) e^{i2(2m+2)\pi x/a} dx = O(\rho(m)). \end{aligned} \right\} \quad (3.25)$$

Thus, from (3.24) and (3.25), $I(m) = O(\rho(m)m^{-1})$. Let us prove that

$$r(m) = \sum_{m_1, m_2} \frac{c_{m_1} c_{m_2} (q h_m, e^{i(2(m-m_1-m_2)+2)\pi x/a})}{m_1(2m+2-m_1)(m_1+m_2)(2m+2-(m_1+m_2))} = O\left(\frac{\rho(m)}{m}\right).$$

Now, by using the relation (2.10), together with (2.12), and taking into account the estimate (3.15), we have

$$|r(m)| \leq C \frac{(\ln|m|)^2}{m^2} \sup_{x \in [0, a]} |h_m(x)| = \rho(m) \frac{(\ln|m|)^2}{m} O\left(\frac{\ln|m|}{m}\right) = O\left(\frac{\rho(m)}{m}\right).$$

where the positive constant C is independent of m . Thus (3.19) holds. Therefore, the first estimate of (3.16) follows from (3.18) and (3.19). In the same way, one can easily obtain the second estimate of (3.16). ■

Now using these lemmas, let us prove the main result of the paper.

Proof of Theorem 2. In view of Lemma 3, substituting the estimates of

$$a(\lambda_{m,j}), a'(\lambda_{m,j}), b(\lambda_{m,j}), b'(\lambda_{m,j}), R(m), R'(m) = O(\rho(m)m^{-1})$$

given by (3.9), (3.12), (3.13), (3.16) in the relations (2.5) and (2.8), we get the following form of the relations

$$\Lambda_{m,j,m} u_{m,j} = c_{2m+2} v_{m,j} + O(\rho(m)m^{-1}), \quad (3.26)$$

$$\Lambda_{m,j,m} v_{m,j} = c_{-2m-2} u_{m,j} + O(\rho(m)m^{-1}) \quad (3.27)$$

for $j = 1, 2$. Again by (2.13), we have, for large m , either $|u_{m,j}| > 1/2$ or $|v_{m,j}| > 1/2$. Without loss of generality we assume that $|u_{m,j}| > 1/2$. Then it follows from both (1.5), (3.26) and (1.5), (3.27) that

$$\Lambda_{m,j,m} \sim c_{2m+2}, \quad (3.28)$$

where the notation $a_m \sim b_m$ is defined in (1.6). This with (1.5), (3.27) and the assumption $|u_{m,j}| > 1/2$ implies that

$$u_{m,j} \sim v_{m,j} \sim 1. \quad (3.29)$$

Thus, first multiplying (3.27) by c_{2m+2} and then using (3.26) in (3.27), we get

$$\Lambda_{m,j,m}(\Lambda_{m,j,m} u_{m,j} + O(\rho(m)m^{-1})) = |c_{2m+2}|^2 u_{m,j} + c_{2m+2} O(\rho(m)m^{-1}),$$

which implies, in view of (3.28)-(3.29), the following equations

$$\Lambda_{m,j,m} = \pm |c_{2m+2}| + O(\rho(m)m^{-1}) \quad (3.30)$$

for $j = 1, 2$. Arguing as in Lemma 4 of [8], let us prove the large periodic eigenvalues are simple. For large m , suppose that there exist two mutually orthogonal eigenfunctions $\Psi_{m,1}(x)$ and $\Psi_{m,2}(x)$ corresponding to $\lambda_{m,1} = \lambda_{m,2}$. Hence, taking into account the expansion (2.11) with $\|h_m\| = O(m^{-1})$ (see (2.12)) for both the eigenfunctions $\Psi_{m,j}(x)$ and then using their orthogonality, we can choose the eigenfunction $\Psi_{m,j}(x)$ such that either $u_{m,j} = 0$ or $v_{m,j} = 0$, which contradicts (3.29).

Finally, for large m , let us prove that each of the simple eigenvalues in (3.30) corresponds to only either the lower sign $-$ or the upper sign $+$, not both. In the first case, we assume that both eigenvalues correspond to the lower sign $-$. Then by (3.26) and (3.30), we get

$$(\Lambda_{m,2,m} - \Lambda_{m,1,m}) u_{m,2} = c_{2m+2} v_{m,2} + |c_{2m+2}| u_{m,2} + O(\rho(m)m^{-1}), \quad (3.31)$$

$$(\Lambda_{m,2,m} - \Lambda_{m,1,m}) u_{m,1} = -c_{2m+2} v_{m,1} - |c_{2m+2}| u_{m,1} + O(\rho(m)m^{-1}). \quad (3.32)$$

Therefore, using the assumption $(\Lambda_{m,2,m} - \Lambda_{m,1,m}) = O(\rho(m)m^{-1})$, multiplying both sides of (3.31) and (3.32) by $v_{m,1}$ and $v_{m,2}$, respectively, and then adding them together, we have, in view of (1.5),

$$u_{m,2}v_{m,1} - u_{m,1}v_{m,2} = o(1). \quad (3.33)$$

Since the eigenfunctions $\Psi_{m,1}$ and $\overline{\Psi_{m,2}}$ of the self-adjoint problem which belong to different eigenvalues $\lambda_{m,1} \neq \lambda_{m,2}$ are orthogonal we have, in view of (2.11),

$$u_{m,2}v_{m,1} + u_{m,1}v_{m,2} = O(m^{-1}).$$

This with (3.33) gives $u_{m,2}v_{m,1} = o(1)$ which contradicts (3.29). The other case, where both eigenvalues correspond to the upper sign $+$, is also impossible. Thus, in (3.30), we may choose the lower sign $-$ for $j = 1$ and the upper sign $+$ for $j = 2$. The theorem is proved. ■

4 Some remarks

1. Applying the same scheme in the paper to the antiperiodic boundary conditions

$$y(0) = -y(a), \quad y'(0) = -y'(a),$$

one can easily obtain similar results in Section 3 for the antiperiodic eigenvalues μ_{2m}, μ_{2m+1} from (1.5), (1.8), (1.9) and (2.1) by replacing $2m + 2$ with $2m + 1$. Then we get $(2m + 1)^2 \pi^2 / a^2$ and c_{2m+1} in (1.3) with the O -term as $O(\rho(m) m^{-1})$.

2. It is well known that the intervals (μ_{2m}, μ_{2m+1}) and $(\lambda_{2m+1}, \lambda_{2m+2})$ are called the instability intervals of the operator (1.1) generated in $L^2(-\infty, \infty)$ by periodicity of q . Using Theorem 2, (1.6) and (1.7), the following asymptotic behaviors of the widths ℓ_{2m+2} of these instability intervals are determined for sufficiently large m . If either condition (1.5) or (1.6) holds, then we have, respectively,

$$\begin{aligned} \ell_{2m+2} &= c_{2m+2}(1 + o(1)), \\ \ell_{2m+2} &= c_{2m+2}(1 + O(m^{-1})), \end{aligned}$$

and if the condition (1.7) holds for sufficiently large m , then we have

$$\ell_{2m+2} = c_{2m+2}(1 + O(\rho(m))),$$

where $\rho(m)$ is defined in (1.4).

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